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BELLMAN–HARRIS PROCESSES WITH HEAVY-TAILED LIFETIMES

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ABSTRACT. We consider a supercritical Bellman–Harris branching process whose lifetime distribution has a regularly varying tail $\overline{G}(t) = L(t)t^{-\beta}$ with $\beta > 0$ .The mean population size $M(t)$ grows like $H_\alpha e^{\alpha t}$, where α is the Malthusian parameter and H_α the Harris constant. We give elementary proofs that under a regularly varying lifetime tail a process with a Malthusian parameter is necessarily supercritical, that the exponentially tilted lifetime law has moments of every order, and that the mean population size is an affine function of the mean number of births. Combined with a renewal density estimate, these facts show that $M(t)$ falls short of $H_\alpha e^{\alpha t}$ by $\overline{G}(t)/(m-1)$, where m is the mean offspring number. A numerical example illustrates the result.

Keywords: Bellman–Harris process; Malthusian parameter; Harris constant; regular variation; renewal equation; mean number of births. MSC (2020): Primary 60J80; Secondary 60K05.

1. INTRODUCTION

A Bellman–Harris process [2] describes a population in which every particle lives a random time and at its death is replaced by a random number of new particles; see [4], [1] and [6] for the general theory. The process starts with one particle of age zero. Its life length τ has distribution function $G(t) = P\{\tau < t\}$ on $T := [0, \ )$ with $G(0+) = 0$, and at death it produces $k \quad N_0$ offspring with probability p_k , independently of its life length. All particles evolve independently by the same rule. We write $\overline{G}(t) := 1 - G(t)$ for the lifetime tail and $m :=$ $_{k\ N} kp_k$ for the mean offspring number.

Let $Z(t)$ be the number of particles alive at time t and $M(t) := EZ(t)$. Conditioning on the life length of the initial particle gives the renewal equation

$$M(t) = \overline{G}(t) + m \ {}_0^t M(t-x)dG(x) \quad \text{for} \quad t \quad T \qquad (1.1)$$

either the initial particle is still alive at t , or it died at some time $x \quad t$ and left on average m independent copies of the process, started afresh at x . For a distribution F on T we write $F(w) :=$ $_0 e^{-wt} dF(t)$ for its Laplace–Stieltjes transform. Suppose that the equation

$$mG(\alpha) = 1 \qquad (1.2)$$

has a real root $\alpha \quad 0$, the Malthusian parameter. Under a nonlattice condition on G , the classical result of Harris [4] states that

$$M(t) \quad H_\alpha e^{\alpha t} \text{ as } t \to \quad \text{, where } H_\alpha = \frac{m-1}{\alpha m \mu_\alpha} \text{ and } \mu_\alpha := m \ {}_0 xe^{-\alpha x} dG(x). \qquad (1.3)$$

The constant H_α is the Harris constant; see also [1].

Relation (1.3+) says nothing about the size of the error. In this note we assume that the lifetime tail is regularly varying,

$$\overline{G}(t) = L(t)t^{-\beta} \quad \text{with} \quad \beta > 0, \qquad (1.4)$$

where $L(t)$ is slowly varying at infinity, that is, $L(\lambda t)/L(t) \to 1$ as $t \to$ for every fixed $\lambda > 0$; see [3]. Long lifetimes of this kind delay reproduction, and one expects the mean to lag behind its Malthusian envelope. We explain, with short proofs, the simple structural facts behind this lag and state the resulting second-order law. The full analysis, including the renewal density estimate used below and the next term of the expansion, is given in [5].

2. LITERATURE REVIEW AND METHODOLOGY

The first observation is that (1.4) fixes the type of the process.

Proposition 1. Assume (1.4) and let (1.2) have a real root α 0. Then $m > 1$ and $\alpha > 0$.

Proof. Let $w < 0$. For every $x > 0$ we have $G(w)$ $e^{|w|x}\overline{G}(x) = e^{|w|x}L(x)x^{-\beta}$, and the right-hand side tends to infinity as $x \to$, because an exponential grows faster than any regularly varying function. Hence $G(w) =$ for $w < 0$, and (1.2) has no negative root. On w 0 the function $G(w)$ decreases strictly from $G(0) = 1$, so $\alpha > 0$ gives $G(\alpha) < 1$ and therefore $m = 1/G(\alpha) > 1$.

Thus a subcritical Malthusian regime is impossible under (1.4): it would require a finite exponential moment of τ. From now on $m > 1$ and $\alpha > 0$. The standard device is the exponential change of measure

$$dG_\alpha(x) := me^{-\alpha x}dG(x), \qquad (2.1)$$

which by (1.2) defines a probability distribution G_α on τ with mean μ_α of (1.3). The tilt removes the heavy tail completely.

Lemma 1. Assume (1.4). Then G_α has finite moments of every order,

$$1 - G_\alpha(t) = o\left(e^{-\alpha t}\overline{G}(t)\right) \quad \text{as} \quad t \to \quad , \qquad (2.2)$$

And

$$\int_0 e^{\alpha y}\left(1 - G_\alpha(y)\right)dy = \frac{m-1}{\alpha}. \qquad (2.3)$$

Proof. The moments are $\int_0 x^j dG_\alpha(x) = m \int_0 x^j e^{-\alpha x}dG(x)$, finite for every j because $\alpha > 0$. For (2.2), integration by parts in $1 - G_\alpha(t) = m \int_t e^{-\alpha y}dG(y)$ and the substitution $y = t + x$ give

$$\frac{1 - G_\alpha(t)}{e^{-\alpha t}\overline{G}(t)} = m \int_0 \alpha e^{-\alpha x}\left(1 - \frac{\overline{G}(t+x)}{\overline{G}(t)}\right)dx.$$

For fixed x the bracket tends to 0 as $t \to$ by regular variation, and it lies in $[0,1]$; dominated convergence gives (2.2). For (2.3), Tonelli's theorem gives

$$\int_0 e^{\alpha y}\left(1-G_\alpha(y)\right)dy = m \int_0 e^{-\alpha x}\int_0^x e^{\alpha y}dy\,dG(x) = \frac{m}{\alpha}\left(1-G(\alpha)\right) = \frac{m-1}{\alpha},$$

the last step by (1.2).

Relation (2.2) shows that the regularly varying index of G is not inherited by G_α. This is why the heavy tail cannot be seen through the moments of the tilted law: its trace appears only at the scale $\overline{G}(t)$ of the original lifetime, and the moment conditions of classical renewal theory never enter, whatever the value of β.

3. RESULTS

Iterating (1.1) expresses $M(t)$ through the convolution powers G^{*n} of G, with $G^{*0}(t)=1$ on τ. Put

$$N(t) := \sum_{n\ N_0} m^n G^{*n}(t). \qquad (3.1)$$

A particle of the n-th generation is born at the sum of the life lengths of its n ancestors, and the expected number of particles in the n-th generation is m^n. Since the numbers of offspring do not depend on the life lengths, the expected number of n-th generation particles born in $[0,t]$ is $m^n G^{*n}(t)$, and $N(t)$ is the expected number of all particles born in $[0,t]$, the initial one included.

Lemma 2. For every t τ,

$$M(t) = \frac{(m-1)N(t)+1}{m}.$$

Proof. The series (3.1) converges. Indeed, by (2.1),
$m^n dG^{*n}(y) = e^{\alpha y} dG_\alpha^{*n}(y)$ and hence $N(t)$ $e^{\alpha t}$ $\sum_{n\ 0} G_\alpha^{*n}(t) <$,

the renewal function of the probability distribution G_α being finite. Separating the term $n=0$, which is the unit point mass at the origin, from the others shows that $N(t) = 1 + m(G * N)(t)$. Hence the function $\int_{[0,t]} \overline{G}(t-y)\,dN(y)$ satisfies (1.1), and since (1.1) has only one solution bounded on finite intervals [4], it equals $M(t)$. Writing $\overline{G}(t-y) = 1 - G(t-y)$, we find

$$M(t) = N(t) - \sum_{n\ 0} m^n G^{*(n+1)}(t) = N(t) - \frac{N(t)-1}{m},$$

which is (3.2).

The sum $\left(N(t)-1\right)/m$ in the proof is the expected number of deaths by time t, and (3.2) records that every particle except the ancestor is born at a death, m at a time on average. The lemma reduces the study of the mean population to the study of births.

4. DISCUSSION

We now assume that G has a bounded completely monotone density $g(t)$, that is, $g(t)$ is a mixture of exponential densities. The canonical example is $\overline{G}(t) = (1+t)^{-\beta}$. Under this

assumption and (1.4) with a non-integer β, the following renewal density estimate is proved in [5]: the function $N(t)$ is absolutely continuous on $(0, \ )$, the difference $\eta(t) := N(t) - e^{\alpha t}/\mu_\alpha$ is integrable on τ with

$$_0 \eta(x)dx = \frac{m}{1-m} + \frac{1}{\alpha\mu_\alpha} \quad \text{and} \quad \eta(t) \quad \frac{m}{(m-1)^2} g(t) \quad \text{as} \quad t \to \quad . \quad (4.1)$$

In words: the mean birth rate exceeds its Malthusian envelope $e^{\alpha t}/\mu_\alpha$ by a fixed multiple of the lifetime density.

Theorem 1. Let (1.4) hold with a non-integer $\beta > 0$, and let G have a bounded completely monotone density. Then

$$N(t) = \frac{e^{\alpha t}}{\alpha\mu_\alpha} - \frac{1}{m-1} - \frac{m\overline{G}(t)}{(m-1)^2} + o\left(\overline{G}(t)\right) \quad \text{as} \quad t \to \quad , \quad (4.2)$$

and

$$M(t) = \mathrm{H}_\alpha e^{\alpha t} - \frac{\overline{G}(t)}{m-1} + o\left(\overline{G}(t)\right) \quad \text{as} \quad t \to \quad . \quad (4.3)$$

Proof. Since $N(0) = 1$,

$$N(t) = 1 + \ _0^t \frac{e^{\alpha x}}{\mu_\alpha} dx + \ _0 \eta(x)dx - \ _t \eta(x)dx = \frac{e^{\alpha t}}{\alpha\mu_\alpha} - \frac{1}{m-1} - \ _t \eta(x)dx \ ,$$

by the first relation in (4.1). Integrating the second relation over $[t, \ )$, which is legitimate because $g(t)$ is positive and integrable, gives $_t \eta(x)dx \quad m\overline{G}(t)/(m-1)^2$ as $t \to$, and (4.2) follows. Inserting (4.2) into (3.2), the leading term becomes $(m-1)e^{\alpha t}/(\alpha m\mu_\alpha) = \mathrm{H}_\alpha e^{\alpha t}$ by (1.3), the constant $-1/(m-1)$ becomes $-1/m$ and cancels against $1/m$, and the tail term becomes $-\overline{G}(t)/(m-1)$. This is (4.3).

Corollary 1. Under the hypotheses of Theorem 1,

$$\frac{M(t)}{H_\alpha e^{\alpha t}} = 1 - \frac{\alpha m\mu_\alpha}{(m-1)^2} e^{-\alpha t}\overline{G}(t) + o\left(e^{-\alpha t}\overline{G}(t)\right) \quad \text{as} \quad t \to \quad .$$

In particular $M(t) < H_\alpha e^{\alpha t}$ for all sufficiently large t.

Proof. Divide (4.3) by $H_\alpha e^{\alpha t}$ and use $1/H_\alpha = \alpha m\mu_\alpha/(m-1)$.

Remark 1. The correction in (4.3) contains neither α nor μ_α: it is the lifetime tail divided by $m-1$. The law holds in the same form for every non-integer $\beta > 0$, including the infinite-mean case $0 < \beta < 1$, because by Lemma 1 the tilted law has moments of every order. The constant $-1/(m-1)$ in (4.2) depends only on the offspring mean.

5. AN EXAMPLE

Let $\overline{G}(t)=(1+t)^{-\beta}$ with a non-integer $\beta>0$. Then (1.4) holds with $L(t)=t/(1+t)^{\beta}$, and the density $g(t)=\beta(1+t)^{-\beta-1}$ is bounded and completely monotone, since

$$(1+t)^{-\beta-1}=\frac{1}{\Gamma(\beta+1)}\int_0^{\infty}\lambda^{\beta}e^{-\lambda}e^{-\lambda t}d\lambda.$$

Integrating by parts, the transform of G is expressed through the upper incomplete Gamma function $\Gamma(-\beta,w)=\int_w^{\infty}x^{-\beta-1}e^{-x}dx$:

$$G(w)=\beta e^{w}w^{\beta}\Gamma(-\beta,w) \quad \text{for} \quad w>0. \tag{5.1}$$

The Malthusian parameter is the root of $mG(\alpha)=1$, and $\mu_{\alpha}=-mG'(\alpha)$ and H_{α} follow from (1.3). Table 1 lists these constants for $m=2$.

TABLE 1. The Malthusian parameter, the tilted mean and the Harris constant for $\overline{G}(t)=(1+t)^{-\beta}$ and $m=2$, computed from (5.1).

β	α	μ_{α}	H_{α}
0.3	0.058128	4.160980	2.067213
0.5	0.187274	1.669886	1.598843
1.5	1.077479	0.392138	1.183374

For $\beta=0.5$ and $m=2$ the expansion (4.3) reads

$$M(t)=1.598843e^{0.187274t}+-(1+t)^{-1/2}+o(t^{-1/2}) \quad \text{as} \quad t\to \quad .$$

Figure 1 compares, for these parameters, the deviation $H_{\alpha}-Y_{\alpha}(t)$ of the normalised mean $Y_{\alpha}(t):=e^{-\alpha t}M(t)$ from its limit with the predicted law $e^{-\alpha t}\overline{G}(t)/(m-1)$. The values of $M(t)$ were obtained by numerical inversion of the Laplace transform $LM(s)=\left(1-G(s)\right)/\left(s\left(1-mG(s)\right)\right)$, which follows from (1.1). The deviation is positive, as Corollary 1 predicts, and the ratio of the computed values to the prediction rises towards 1, from $0,76$ at $t=3$ to $0,93$ at $t=25$.

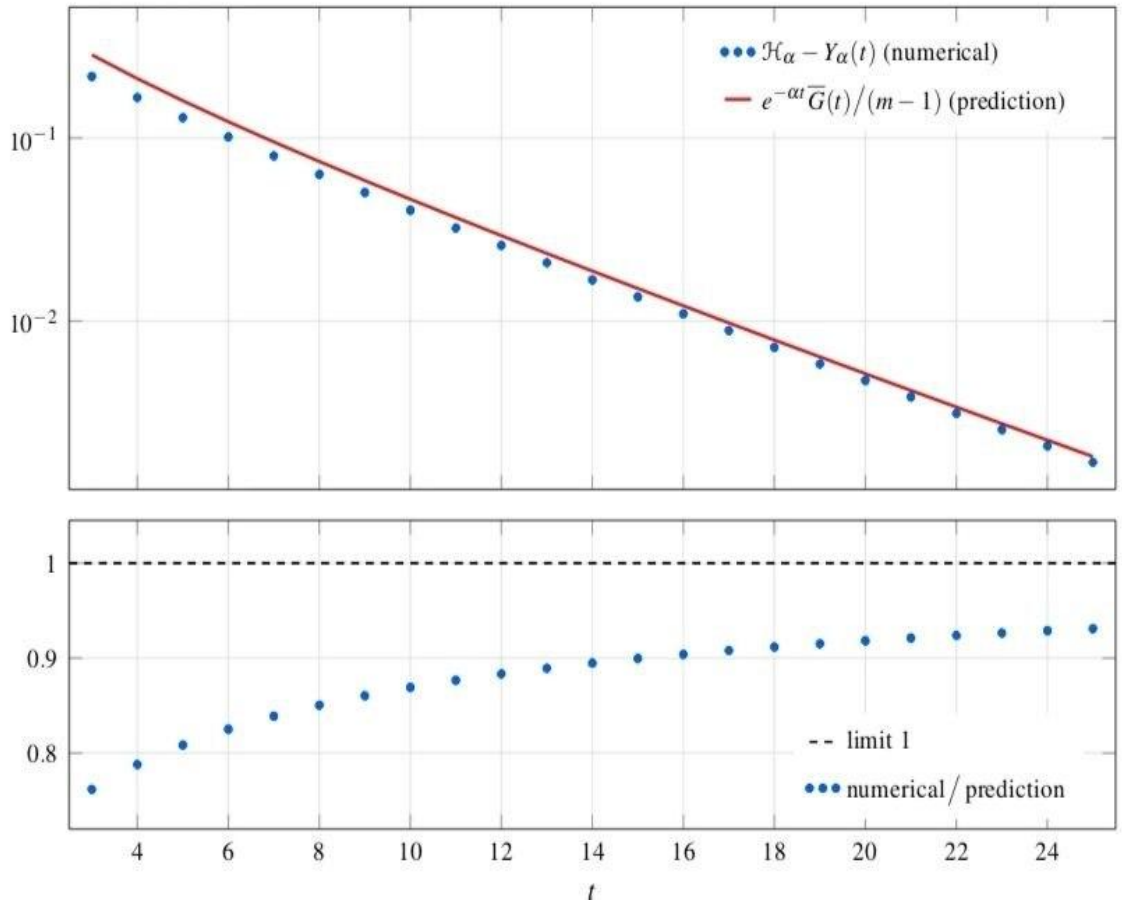


FIGURE 1. The deviation $H_\alpha - Y_\alpha(t)$ on a logarithmic scale against the prediction of Theorem 1, for $\overline{G}(t) = (1+t)^{-1/2}$ and $m = 2$. The lower panel shows their ratio, which rises towards the limit 1.

6. CONCLUSION

In this paper, we have investigated the asymptotic behavior of supercritical

Bellman–Harris branching processes with regularly varying heavy-tailed lifetime distributions. We have established several fundamental properties of such processes and derived a second-order asymptotic expansion for the mean population size. First, it has been proved that the existence of a Malthusian parameter under a regularly varying lifetime distribution necessarily implies supercriticality and the positivity of the Malthusian parameter. Furthermore, we have shown that the exponentially tilted lifetime distribution possesses finite moments of all orders, despite the heavy-tailed nature of the original lifetime distribution. An important structural result establishes an explicit affine relationship between the mean population size and the expected number of births.

Under the additional assumption that the lifetime distribution has a bounded completely monotone density, we have obtained a second-order asymptotic formula demonstrating that the mean population size approaches its Malthusian exponential growth envelope from below. The leading correction term is determined by the lifetime tail and the mean offspring number, and its form remains valid for all non-integer values of the regular variation index, including the infinite-mean lifetime case.

Finally, the numerical example confirms the theoretical asymptotic result and illustrates the convergence of the normalized deviation to the predicted limit. These findings provide a more

precise description of the influence of heavy-tailed lifetimes on population growth in Bellman–Harris branching processes.

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